

FULL CHARACTERIZATION OF EMBEDDING RELATIONS BETWEEN α -MODULATION SPACES

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ABSTRACT. In this paper, we consider the embedding relations between any two α -modulation spaces. Based on an observation that the α -modulation space with smaller α can be regarded as a corresponding α -modulation space with larger α , we give a complete characterization of the Fourier multipliers between α -modulation spaces with different α . Then we establish a full version of optimal embedding relations between α -modulation spaces.

1. INTRODUCTION

As we know, the decomposition method on frequency plays an important role in the study of function spaces and their applications. Among many others, two basic types of decomposition are used most frequently. One is the uniform decomposition and the other is the dyadic decomposition. The corresponding function spaces associated with these two decompositions are the modulation spaces and the Besov spaces, respectively.

The modulation spaces, introduced by Feichtinger [7] in 1983, was firstly defined by the short-time Fourier transform. We refer the reader to see [6, 7, 11, 22] for some basic properties and applications of the modulation spaces, as well as their historical developments. Particularly, there is an equivalent definition of modulation space using the uniform decomposition on the frequency plane. On the other hand, it is well known that the Besov space $B_{p,q}^s$ (see [20]), constructed by the dyadic decomposition on frequency plane, is also a popular working frame in the fields of harmonic analysis and partial differential equations.

In the eighties of last century, a so-called α -covering on the frequency plane was found in [4, 5]. This covering is an intermediate decomposition method between the uniform decomposition and the dyadic decomposition. Applying the α -covering to the frequency plane, Gröbner [10] introduced the α -modulation spaces $M_{p,q}^{s,\alpha}$ with respect to the parameters $\alpha \in [0, 1]$. The space $M_{p,q}^{s,\alpha}$ coincides with the modulation space $M_{p,q}^s$ when $\alpha = 0$, and, in some sense, it coincides with the Besov space $B_{p,q}^s$ when $\alpha = 1$ (see [10]). So, for the sake of convenience, we can view the Besov space as a special α -modulation space and use $M_{p,q}^{s,1}$ to denote the inhomogeneous Besov space $B_{p,q}^s$.

In the last ten years, the α -modulation space received extensive attention. Its many algebraic properties and geometric characterizations were discovered, and many of its applications were established. For the details, the reader can see [1, 9]

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for the Banach frames of α -modulation spaces, [2, 3, 24, 25] for the boundedness of certain operators in the frame of α -modulation spaces. We also refer the reader to [13, 14] for the study of some fundamental properties about α -modulation spaces. Among many features of the α -modulation spaces, a particularly interesting subject is the embedding between two different α -modulation spaces. As an analogy of the Sobolev embedding on the Lebesgue spaces, the embedding among the different α -modulation spaces plays a notable role in the study of partial differential equations and in theory of function spaces. The research for embedding relation between α_1 -modulation and α_2 -modulation spaces goes back to Gröbner's thesis [10], where he considered the case $1 \leq p, q \leq \infty$. More sharp results were established in [19], in which Toft and Wahlberg obtained some partial sufficient conditions, as well as some partial necessary conditions for such embedding between the α -modulation spaces (One also can see [17, 18, 23] for the embedding between modulation and Besov spaces, and [15] for the embedding between modulation spaces and Sobolev spaces). Especially, the embedding relations between $M_{p,q}^{s_1,\alpha_1}$ and $M_{p,q}^{s_2,\alpha_2}$ has been completely determined by Wang and Han in [14]. Their result can be stated in the following proposition.

Proposition 1.1 ([14], **Theorem 4.1 and Theorem 4.2**). *Let $0 < p, q \leq \infty$, $s_i \in \mathbb{R}$, $\alpha_i \in [0, 1]$ for $i = 1, 2$. Then*

$$M_{p,q}^{s_1,\alpha_1} \subset M_{p,q}^{s_2,\alpha_2} \quad (1.1)$$

holds if and only if

$$s_2 + 0 \vee [n(\alpha_2 - \alpha_1)(1/p - 1/q)] \vee [n(\alpha_2 - \alpha_1)(1 - 1/p - 1/q)] \leq s_1,$$

where the notation $a \vee b$ denotes the maximum between a and b .

We notice that all the previous results about the embedding relation between $M_{p_1,q_1}^{s_1,\alpha_1}$ and $M_{p_2,q_2}^{s_2,\alpha_2}$ concern only some special p_1, p_2, q_1, q_2 . Hence, it will be of great interest if we establish the sharp embedding $M_{p_1,q_1}^{s_1,\alpha_1} \subseteq M_{p_2,q_2}^{s_2,\alpha_2}$ in full ranges $0 < p_i, q_i \leq \infty$, $s_i \in \mathbb{R}$, and $\alpha_i \in [0, 1]$ for $i = 1, 2$. However, the complexity of the methods used in previous works make us quite difficult to adopt these methods to treat more general situations. A different and more efficient method might be necessarily introduced. Based upon these motivation and observation, the main goal of this paper is to seek a new method to give a complete characterization of the embedding relation between any two α -modulation spaces. The following theorem is our main result.

Theorem 1.2 (Sharpness of embedding). *Let $0 < p_i, q_i \leq \infty$, $s_i \in \mathbb{R}$, $\alpha_i \in [0, 1]$ for $i = 1, 2$. Then*

$$M_{p_1,q_1}^{s_1,\alpha_1} \subseteq M_{p_2,q_2}^{s_2,\alpha_2} \quad (1.2)$$

if and only if

$$\begin{cases} \frac{1}{p_2} \leq \frac{1}{p_1} \\ s_2 + R(\mathbf{p}, \mathbf{q}, \alpha_1, \alpha_2) \leq s_1 \\ \frac{1}{q_2} \leq \frac{1}{q_1} \end{cases}, \quad (1.3)$$

or

$$\begin{cases} \frac{1}{p_2} \leq \frac{1}{p_1} \\ s_2 + R(\mathbf{p}, \mathbf{q}, \alpha_1, \alpha_2) + \frac{n(1-\alpha_1 \vee \alpha_2)}{q_2} < s_1 + \frac{n(1-\alpha_1 \vee \alpha_2)}{q_1} \\ \frac{1}{q_2} > \frac{1}{q_1} \end{cases} \quad (1.4)$$

holds. Here, we denote

$$R(\mathbf{p}, \mathbf{q}; \alpha_1, \alpha_2) = \begin{cases} \left[n\alpha_1 \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \right] \vee \left[n\alpha_2 \left(1 - \frac{1}{p_2} \right) - n\alpha_1 \left(1 - \frac{1}{p_1} \right) - n(\alpha_2 - \alpha_1) \frac{1}{q_1} \right] \\ \quad \vee \left[n(\alpha_2 - \alpha_1) \left(\frac{1}{p_2} - \frac{1}{q_1} \right) + n\alpha_1 \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \right], & \text{if } \alpha_1 \leq \alpha_2, \\ \left[n\alpha_2 \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \right] \vee \left[n\alpha_2 \left(1 - \frac{1}{p_2} \right) - n\alpha_1 \left(1 - \frac{1}{p_1} \right) - n(\alpha_2 - \alpha_1) \frac{1}{q_2} \right] \\ \quad \vee \left[n(\alpha_1 - \alpha_2) \left(\frac{1}{q_2} - \frac{1}{p_1} \right) + n\alpha_2 \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \right], & \text{if } \alpha_1 > \alpha_2, \end{cases}$$

where $\mathbf{p} = (p_1, p_2)$, $\mathbf{q} = (q_1, q_2)$.

Remark 1.3. Obviously, our result gives a complete characterization of the embedding relations between any two α -modulation spaces, which is an essential improvement and extension to Theorem A. We also remark that a similar embedding problem is studied by Voigtlaender in [21], in which a more general framework, the decomposition spaces, is considered. However, our work has its own interesting.

As we mentioned previously, neither the method in [19], nor the proof used in proving Proposition 1.1 seems adoptable in our proof on this more general situation. Thus, we will use a new and more efficient approach to achieve our target. Below, we outline the strategy of our proof for Theorem 1.2.

Firstly, in Section 3, we will give a characterization of Fourier multipliers between α -modulation spaces by means of the corresponding Wiener amalgam spaces. To be precise, in Theorem 3.2, we show that a Fourier multiplier T_m is bounded from $M_{p_1, q_1}^{s_1, \alpha_1}$ to $M_{p_2, q_2}^{s_2, \alpha_2}$ if and only if the norm sequence

$$\{\mathfrak{S}_k\} = \left\{ \left\| \square_k^{\alpha_1 \vee \alpha_2} T_m \right\|_{M_{p_1, q_1}^{0, \alpha_1} \rightarrow M_{p_2, q_2}^{0, \alpha_2}} \right\}_{k \in \mathbb{Z}^n}$$

is a pointwise multiplier from the sequence space $l_{q_1}^{s_1, \alpha_1 \vee \alpha_2}$ to the sequence space $l_{q_2}^{s_2, \alpha_1 \vee \alpha_2}$, where $\{\square_k^{\alpha_1 \vee \alpha_2} T_m\}$ is the sequence of localizations of T_m based on the α -covering on the frequency plane. Since the embedding $M_{p_1, q_1}^{s_1, \alpha_1} \subset M_{p_2, q_2}^{s_2, \alpha_2}$ can be viewed as the boundedness of the identity operator mapping from $M_{p_1, q_1}^{s_1, \alpha_1}$ to $M_{p_2, q_2}^{s_2, \alpha_2}$, it is reduced from Theorem 3.2 that the embedding relation $M_{p_1, q_1}^{s_1, \alpha_1} \subset M_{p_2, q_2}^{s_2, \alpha_2}$ holds if and only if the norm sequence

$$\{\mathfrak{R}_k\}_{k \in \mathbb{Z}^n} = \left\{ \left\| \square_k^{\alpha_1 \vee \alpha_2} \right\|_{M_{p_1, q_1}^{0, \alpha_1} \rightarrow M_{p_2, q_2}^{0, \alpha_2}} \right\}_{k \in \mathbb{Z}^n}$$

is a pointwise multiplier from the sequence space $l_{q_1}^{s_1, \alpha_1 \vee \alpha_2}$ to the sequence space $l_{q_2}^{s_2, \alpha_1 \vee \alpha_2}$. Thus, to prove Theorem 1.2, it suffices to find the precise asymptotic estimate of the sequence $\{\mathfrak{R}_k\}_{k \in \mathbb{Z}^n}$. This task is quite technical, and it will be finished in Lemma 4.1 for the case $q_1 = q_2 = q$ with the help of complex interpolations and a constructive proof. The asymptotic estimate for the general case can be finally obtained by invoking Lemma 4.1 and some technical treatments. By the asymptotic estimates of local operators between α -modulation spaces, we can verify the sufficient and necessary conditions simultaneously, which completes the proof of Theorem 1.2.

We explain the organization of this paper. In Section 2, we give some definitions of function spaces treated in this paper. We also collect some basic properties used in our proof. We establish the framework for the proof of Theorem 1.2 in Section 3. In Section 3, we first use Proposition 3.1 to illustrate our viewpoint about

α -modulation spaces with different α . By the spirit of this viewpoint, we give a complete characterization of Fourier multipliers between any two α -modulation spaces. Then we obtain the reduction of Theorem 1.2. In Section 4, we establish some asymptotic estimates of local operators between α -modulation spaces, which is the quantity part for the proof of Theorem 1.2. Based on the preparatory work in Section 3 and Section 4, we will complete the proof of our main theorem in Section 5. Also in Section 5, we will make some comments about this paper.

2. PRELIMINARIES

We recall some notations. Let C be a positive constant that may depend on n, p_i, q_i, s_i, α . The notation $X \lesssim Y$ denotes the statement that $X \leq CY$, the notation $X \sim Y$ means the statement $X \lesssim Y \lesssim X$, and the notation $X \simeq Y$ denotes the statement $X = CY$. For a multi-index $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$, we denote $|k|_\infty := \max_{i=1,2,\dots,n} |k_i|$, and $\langle k \rangle := (1 + |k|^2)^{\frac{1}{2}}$.

Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ be the Schwartz space and $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distributions. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \quad \mathcal{F}^{-1}f(x) = \hat{f}(-x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

We recall some definitions of the function spaces treated in this paper. For the convenience of doing calculation, we give the definition of α -modulation spaces based on the decomposition method. This definition is equivalent to the one in [10]. Suppose that $c > 0$ and $C > 0$ are two appropriate constants. Choose a sequence of Schwartz functions $\{\eta_k^\alpha\}_{k \in \mathbb{Z}^n}$ satisfying

$$\begin{cases} |\eta_k^\alpha(\xi)| \geq 1, \text{ if } |\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k| < c \langle k \rangle^{\frac{\alpha}{1-\alpha}}; \\ \text{supp } \eta_k^\alpha \subset \{\xi : |\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k| < C \langle k \rangle^{\frac{\alpha}{1-\alpha}}\}; \\ \sum_{k \in \mathbb{Z}^n} \eta_k^\alpha(\xi) \equiv 1, \forall \xi \in \mathbb{R}^n; \\ |\partial^\gamma \eta_k^\alpha(\xi)| \leq C_\alpha \langle k \rangle^{-\frac{|\gamma|}{1-\alpha}}, \forall \xi \in \mathbb{R}^n, \gamma \in (\mathbb{Z}^+ \cup \{0\})^n. \end{cases} \quad (2.1)$$

Then $\{\eta_k^\alpha(\xi)\}_{k \in \mathbb{Z}^n}$ constitutes a smooth decomposition of \mathbb{R}^n . The frequency decomposition operators associated with the above function sequence can be defined by

$$\square_k^\alpha := \mathcal{F}^{-1} \eta_k^\alpha \mathcal{F} \quad (2.2)$$

for $k \in \mathbb{Z}^n$. Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$, $\alpha \in [0, 1)$. The α -modulation space associated with the above decomposition is defined by

$$M_{p,q}^{s,\alpha}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} = \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{sq}{1-\alpha}} \|\square_k^\alpha f\|_p^q \right)^{\frac{1}{q}} < \infty\} \quad (2.3)$$

with the usual modifications when $q = \infty$. For simplicity, we denote $M_{p,q}^s = M_{p,q}^{s,0}$ and $\eta_k(\xi) = \eta_k^0(\xi)$.

Remark 2.1. We recall that the above definition is independent of the choice of exact η_k^α (see [14]). Also, for sufficiently small $\delta > 0$, one can construct a function sequence $\{\eta_k^\alpha(\xi)\}_{k \in \mathbb{Z}^n}$ such that $\eta_k^\alpha(\xi) = 1$ and $\eta_k^\alpha(\xi) \eta_l^\alpha(\xi) = 0$ if $k \neq l$, when ξ lies in the ball $B(\langle k \rangle^{\frac{1-\alpha}{\alpha}} k, \langle k \rangle^{\frac{1-\alpha}{\alpha}} \delta)$ (see [1, 9, 12]).

For $\alpha \in [0, 1)$, we set

$$\Lambda_k^\alpha = \{l \in \mathbb{Z}^n : \square_l^\alpha \circ \square_k^\alpha \neq 0\} \quad (2.4)$$

and

$$\Lambda_k^{\alpha,*} = \{l \in \mathbb{Z}^n : \square_l^\alpha \circ \square_m^\alpha \neq 0 \text{ for some } m \in \Lambda_k^\alpha\}. \quad (2.5)$$

Denote

$$\square_k^{\alpha,*} = \sum_{l \in \Lambda_k^\alpha} \square_l^\alpha, \quad \eta_k^{\alpha,*} = \sum_{l \in \Lambda_k^\alpha} \eta_l^\alpha. \quad (2.6)$$

For $\alpha_1, \alpha_2 \in (0, 1)$, $k \in \mathbb{Z}^n$, we denote

$$\Gamma_k^{\alpha_1, \alpha_2} = \{l \in \mathbb{Z}^n : \square_k^{\alpha_2} \circ \square_l^{\alpha_1} \neq 0\}, \quad \widehat{\Gamma_k^{\alpha_1, \alpha_2}} = \{l \in \mathbb{Z}^n : \square_k^{\alpha_2} \circ \square_l^{\alpha_1} = \square_l^{\alpha_1}\}. \quad (2.7)$$

To define the Besov space, we introduce the dyadic decomposition of \mathbb{R}^n . Let $\varphi(\xi)$ be a smooth bump function supported in the ball $\{\xi : |\xi| < \frac{3}{2}\}$ and be equal to 1 on the ball $\{\xi : |\xi| \leq \frac{4}{3}\}$. For integers $j \in \mathbb{Z}$, we define the Littlewood-Paley operators

$$\begin{aligned} \widehat{\Delta_j f} &= (\varphi(2^{-j}\xi) - \varphi(2^{-j+1}\xi)) \widehat{f}(\xi), \quad j \geq 0, \\ \widehat{\Delta_0 f} &= \varphi(\xi) \widehat{f}(\xi). \end{aligned} \quad (2.8)$$

Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'$, set

$$\|f\|_{B_{p,q}^s} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\Delta_j f\|_{L^p}^q \right)^{1/q}. \quad (2.9)$$

The (inhomogeneous) Besov space is the space of all tempered distributions f for which the quantity $\|f\|_{B_{p,q}^s}$ is finite.

Suppose $0 < q \leq \infty$, $s \in \mathbb{R}$ and $\alpha \in [0, 1)$. Let $\{\lambda_k\}_{k \in \mathbb{Z}^n}$ denote a sequence of complex numbers. Set

$$\|\{\lambda_k\}\|_{l_q^{s,\alpha}} = \begin{cases} \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{sq}{1-\alpha}} |\lambda_k|^q \right)^{\frac{1}{q}} & \text{if } 0 < q < \infty, \\ \sup_{k \in \mathbb{Z}^n} (\langle k \rangle^{\frac{s}{1-\alpha}} |\lambda_k|) & \text{if } q = \infty. \end{cases} \quad (2.10)$$

We use $l_q^{s,\alpha}$ to denote the set of all sequences $\{\lambda_k\}_{k \in \mathbb{Z}^n}$ such that $\|\{\lambda_k\}\|_{l_q^{s,\alpha}} < \infty$.

Similarly, we use $l_q^{s,1}$ to denote the set of all sequences $\{\lambda_j\}_{j \in \mathbb{N}}$ such that

$$\|\{\lambda_j\}\|_{l_q^{s,1}} = \begin{cases} \left(\sum_{j \in \mathbb{N}} 2^{js} |a_j|^q \right)^{\frac{1}{q}} & \text{if } 0 < q < \infty, \\ \sup_{j \in \mathbb{N}} (2^{js} |a_j|) & \text{if } q = \infty \end{cases} \quad (2.11)$$

is finite.

Now, we define the space of pointwise multipliers between sequence spaces. For $\alpha \in [0, 1)$, we set

$$\mathcal{M}_p(l_{q_1}^{s_1,\alpha}, l_{q_2}^{s_2,\alpha}) = \left\{ \{a_k\}_{k \in \mathbb{Z}^n} : \|\{a_k \lambda_k\}\|_{l_{q_2}^{s_2,\alpha}} \lesssim \|\{\lambda_k\}\|_{l_{q_1}^{s_1,\alpha}} \text{ for all } \{\lambda_k\} \in l_{q_1}^{s_1,\alpha} \right\}. \quad (2.12)$$

For $\alpha = 1$, we set

$$\mathcal{M}_p(l_{q_1}^{s_1,1}, l_{q_2}^{s_2,1}) = \left\{ \{a_j\}_{j \in \mathbb{N}} : \|\{a_j \lambda_j\}\|_{l_{q_2}^{s_2,1}} \lesssim \|\{\lambda_j\}\|_{l_{q_1}^{s_1,1}} \text{ for all } \{\lambda_j\} \in l_{q_1}^{s_1,1} \right\}. \quad (2.13)$$

Denote

$$\|\{a_k\}\|_{\mathcal{M}_p(l_{q_1}^{s_1,\alpha}, l_{q_2}^{s_2,\alpha})} = \|\{a_k\}\|_{l_{q_1}^{s_1,\alpha} \rightarrow l_{q_2}^{s_2,\alpha}} = \sup_{\|\{\lambda_k\}\|_{l_{q_1}^{s_1,\alpha}=1}} \|\{a_k \lambda_k\}\|_{l_{q_2}^{s_2,\alpha}}. \quad (2.14)$$

We list the following lemmas which will be used frequently in our proof.

Lemma 2.2 (Embedding of L^p with Fourier compact support, [20]). *Let $0 < p_1 \leq p_2 \leq \infty$ and assume $\text{supp } \hat{f} \subseteq B(0, R)$. We have*

$$\|f\|_{L^{p_2}} \leq CR^{n(\frac{1}{p_1} - \frac{1}{p_2})} \|f\|_{L^{p_1}}, \quad (2.15)$$

where C is independent of f .

Lemma 2.3 (Convolution in L^p with $p < 1$, [20]). *Let $0 < p < 1$ and $L_{B(x_0, R)}^p = \{f \in L^p(\mathbb{R}^n) : \text{supp } \hat{f} \subseteq B(x_0, R)\}$, where $B(x_0, R) = \{x : |x - x_0| \leq R\}$. Suppose $f, g \in L_{B(x_0, R)}^p$. Then there exists a constant $C > 0$ which is independent of x_0 and $R > 0$ such that*

$$\|f * g\|_p \leq CR^{n(1/p-1)} \|f\|_p \|g\|_p.$$

Lemma 2.4 (see [14]). *Let $0 < p_i, q_i \leq \infty$, $s_i \in \mathbb{R}$ for $i = 1, 2$ and $\alpha \in [0, 1]$. Then we have*

$$[M_{p_1, q_1}^{s_1, \alpha}, M_{p_2, q_2}^{s_2, \alpha}]_\theta = M_{p_\theta, q_\theta}^{s_\theta, \alpha} \quad (2.16)$$

for $\theta \in (0, 1)$, where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad s_\theta = (1-\theta)s_1 + \theta s_2.$$

Remark 2.5. In the rest of this paper, for simplicity in the notation, we denote

$$M_i^{s_i} = M_{p_i, q_i}^{s_i, \alpha_i}, \quad M_i = M_{p_i, q_i}^{0, \alpha_i} \quad (2.17)$$

for $i = 1, 2$, when no confusion is possible.

3. FOURIER MULTIPLIERS ON α -MODULATION SPACES

In this section, we display some propositions to explain the framework for the proof of Theorem 1.2. And each of these propositions also has its independent significance.

Firstly, we recall the previous study of Fourier multiplier on frequency decomposition spaces. In Feichtinger-Narimani [8], the authors study the Fourier multiplier between M_{p_1, q_1} and M_{p_2, q_2} , where $1 \leq p_i, q_i \leq \infty$ for $i = 1, 2$, one can also see Feichtinger-Gröbner [4] for a general result in the frame of Banach space (with same decomposition). Recently, in order to study the behavior of unimodular multiplier on α -modulation spaces, in [24], we establish a corresponding result between $M_{p_1, q_1}^{s_1, \alpha}$ and $M_{p_2, q_2}^{s_2, \alpha}$, where $1 \leq p_i, q_i \leq \infty$, $s_i \in \mathbb{R}$.

In this section, we give a full characterization of Fourier multipliers between any two α -modulation spaces, which extends all the previous results. Especially, our theorem covers the case that $\alpha_1 \neq \alpha_2$ and $s_1 \neq s_2$, which allows the different decompositions and different potentials. Our theorem also covers the Quasi-Banach case ($p < 1$ or $q < 1$), which is not contained in the previous results.

Our method is based on the observation that an α -modulation space with smaller α can be regarded as a corresponding α -modulation space with larger α . The following proposition demonstrates this viewpoint.

Proposition 3.1. *Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$, $\alpha_i \in [0, 1]$ and $\alpha_1 \leq \alpha_2$. We have*

$$\|f\|_{M_{p,q}^{s,\alpha_1}} \sim \begin{cases} \|\{\|\square_k^{\alpha_2} f\|_{M_{p,q}^{0,\alpha_1}}\} l_q^{s,\alpha_2}\|, & \text{if } \alpha_2 < 1 \\ \|\{\|\Delta_j f\|_{M_{p,q}^{0,\alpha_1}}\} l_q^{s,1}\|, & \text{if } \alpha_2 = 1. \end{cases} \quad (3.1)$$

Proof. We only state the proof for $\alpha_2 < 1$, $q < \infty$, since the proof for the other cases shares the same idea. Firstly, by the definition we have

$$\|\square_k^{\alpha_2} f\|_{M_{p,q}^{0,\alpha_1}} = \left(\sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} \square_k^{\alpha_2} f\|_{L^p}^q \right)^{1/q}. \quad (3.2)$$

Using Lemma 2.3 or the Young's inequality, we deduce

$$\|\square_l^{\alpha_1} \square_k^{\alpha_2} f\|_{L^p} \lesssim \langle k \rangle^{\frac{\alpha_2 n}{1-\alpha_2}(\frac{1}{p\wedge 1}-1)} \|\mathcal{F}^{-1} \eta_k^{\alpha_2}\|_{L^{p\wedge 1}} \|\square_l^{\alpha_1} f\|_{L^p} \lesssim \|\square_l^{\alpha_1} f\|_{L^p}, \quad (3.3)$$

it follows that

$$\|\square_k^{\alpha_2} f\|_{M_{p,q}^{0,\alpha_1}} \lesssim \left(\sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^p}^q \right)^{1/q}. \quad (3.4)$$

Observing $|\Gamma_l^{\alpha_2, \alpha_1}| \lesssim 1$ for $\alpha_1 \leq \alpha_2$, we deduce

$$\begin{aligned} \|\{\|\square_k^{\alpha_2} f\|_{M_{p,q}^{0,\alpha_1}}\} l_q^{s,\alpha_2}\| &\lesssim \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{sq}{1-\alpha_2}} \sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^p}^q \right)^{1/q} \\ &\sim \left(\sum_{l \in \mathbb{Z}^n} \sum_{k \in \Gamma_l^{\alpha_2, \alpha_1}} \langle l \rangle^{\frac{sq}{1-\alpha_1}} \|\square_l^{\alpha_1} f\|_{L^p}^q \right)^{1/q} \\ &\lesssim \left(\sum_{l \in \mathbb{Z}^n} \langle l \rangle^{\frac{sq}{1-\alpha_1}} \|\square_l^{\alpha_1} f\|_{L^p}^q \right)^{1/q} \lesssim \|f\|_{M_{p,q}^{s,\alpha_1}}. \end{aligned} \quad (3.5)$$

On the other hand, we have

$$\|\square_l^{\alpha_1} f\|_{L^p} = \left\| \sum_{k \in \Gamma_l^{\alpha_2, \alpha_1}} \square_k^{\alpha_2} \square_l^{\alpha_1} f \right\|_{L^p}. \quad (3.6)$$

Observing $|\Gamma_l^{\alpha_2, \alpha_1}| \lesssim 1$, we deduce

$$\left\| \sum_{k \in \Gamma_l^{\alpha_2, \alpha_1}} \square_k^{\alpha_2} \square_l^{\alpha_1} f \right\|_{L^p}^q \lesssim \sum_{k \in \Gamma_l^{\alpha_2, \alpha_1}} \|\square_k^{\alpha_2} \square_l^{\alpha_1} f\|_{L^p}^q. \quad (3.7)$$

This leads to

$$\begin{aligned} \|f\|_{M_{p,q}^{s,\alpha_1}} &= \left(\sum_{l \in \mathbb{Z}^n} \langle l \rangle^{\frac{sq}{1-\alpha_1}} \|\square_l^{\alpha_1} f\|_{L^p}^q \right)^{1/q} \\ &\lesssim \left(\sum_{l \in \mathbb{Z}^n} \langle l \rangle^{\frac{sq}{1-\alpha_1}} \sum_{k \in \Gamma_l^{\alpha_2, \alpha_1}} \|\square_k^{\alpha_2} \square_l^{\alpha_1} f\|_{L^p}^q \right)^{1/q} \\ &\sim \left(\sum_{l \in \mathbb{Z}^n} \sum_{k \in \Gamma_l^{\alpha_2, \alpha_1}} \langle k \rangle^{\frac{sq}{1-\alpha_2}} \|\square_k^{\alpha_2} \square_l^{\alpha_1} f\|_{L^p}^q \right)^{1/q}. \end{aligned} \quad (3.8)$$

By exchanging the summation order, we deduce that

$$\begin{aligned}
\left(\sum_{l \in \mathbb{Z}^n} \sum_{k \in \Gamma_l^{\alpha_2, \alpha_1}} \langle k \rangle^{\frac{sq}{1-\alpha_2}} \|\square_k^{\alpha_2} \square_l^{\alpha_1} f\|_{L^p}^q \right)^{1/q} &= \left(\sum_{k \in \mathbb{Z}^n} \sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \langle k \rangle^{\frac{sq}{1-\alpha_2}} \|\square_k^{\alpha_2} \square_l^{\alpha_1} f\|_{L^p}^q \right)^{1/q} \\
&= \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{sq}{1-\alpha_2}} \sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} \square_k^{\alpha_2} f\|_{L^p}^q \right)^{1/q} \\
&= \left\| \{ \|\square_k^{\alpha_2} f\|_{M_{p,q}^{0,\alpha_1}} \} | l_q^{s,\alpha_2} \right\|.
\end{aligned} \tag{3.9}$$

Combining with (3.8) and (3.9), we obtain our conclusion. \square

By the spirit of the above proposition, we are able to give a full characterization of Fourier multipliers between any two α -modulation spaces in the following Theorem 3.2. Our theorem extends the known result in [8], where the authors only consider the special case $\alpha_1 = \alpha_2 = 0$. We would like to remark that our elementary method allows us to handle more general cases in Theorem 3.2, even including the Quasi-Banach case.

A tempered distribution m is called a Fourier multiplier from $M_{p_1,q_1}^{s_1,\alpha_1}$ to $M_{p_2,q_2}^{s_2,\alpha_2}$, if there exists a constant $C > 0$ such that

$$\|T_m(f)\|_{M_{p_2,q_2}^{s_2,\alpha_2}} \leq C \|f\|_{M_{p_1,q_1}^{s_1,\alpha_1}}$$

for all f in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, where

$$T_m f = m(D)f = \mathcal{F}^{-1}(m \mathcal{F} f)$$

is the Fourier multiplier operator associated with m , and m is called the symbol or multiplier of T_m . Let $\mathcal{M}_{\mathcal{F}}(X, Y)$ denote the set of all symbols such that the corresponding Fourier multipliers are bounded from X to Y . We set

$$\|m\|_{\mathcal{M}_{\mathcal{F}}(X, Y)} = \|T_m\|_{X \rightarrow Y} = \sup\{\|m(D)f\|_Y : f \in \mathcal{S}(\mathbb{R}^n), \|f\|_X = 1\}, \tag{3.10}$$

where X and Y denote certain α -modulation spaces.

For the sake of convenience, we define some exact Wiener amalgam spaces. For $m \in \mathcal{S}'$, $0 < p_i, q_i \leq \infty$, $s_i \in \mathbb{R}$, $\alpha_i \in [0, 1]$ for $i = 1, 2$, we denote

$$\|m\|_{W^\alpha(\mathcal{M}_{\mathcal{F}}(M_{p_1,q_1}^{0,\alpha_1}, M_{p_2,q_2}^{0,\alpha_2}), \mathcal{M}_p(l_{q_1}^{s_1}, l_{q_2}^{s_2}))} = \|\{\|\square_k^\alpha T_m\|_{M_{p_1,q_1}^{0,\alpha_1} \rightarrow M_{p_2,q_2}^{0,\alpha_2}}\} | l_{q_1}^{s_1,\alpha} \rightarrow l_{q_2}^{s_2,\alpha}\| \tag{3.11}$$

for $\alpha \in [0, 1]$. Similarly, we denote

$$\|m\|_{W^1(\mathcal{M}_{\mathcal{F}}(M_{p_1,q_1}^{0,\alpha_1}, M_{p_2,q_2}^{0,\alpha_2}), \mathcal{M}_p(l_{q_1}^{s_1}, l_{q_2}^{s_2}))} = \|\{\|\Delta_j T_m\|_{M_{p_1,q_1}^{0,\alpha_1} \rightarrow M_{p_2,q_2}^{0,\alpha_2}}\} | l_{q_1}^{s_1,1} \rightarrow l_{q_2}^{s_2,1}\|. \tag{3.12}$$

Theorem 3.2 (Characterization of Fourier multipliers on α -modulation spaces). *Let $0 < p_i, q_i \leq \infty$, $s_i \in \mathbb{R}$, $\alpha_i \in [0, 1]$ for $i = 1, 2$. Then we have*

$$\mathcal{M}_{\mathcal{F}}(M_{p_1,q_1}^{s_1,\alpha_1}, M_{p_2,q_2}^{s_2,\alpha_2}) = W^{\alpha_1 \vee \alpha_2}(\mathcal{M}_{\mathcal{F}}(M_{p_1,q_1}^{0,\alpha_1}, M_{p_2,q_2}^{0,\alpha_2}), \mathcal{M}_p(l_{q_1}^{s_1}, l_{q_2}^{s_2})). \tag{3.13}$$

Proof. We only give the proof for $\alpha_1, \alpha_2 < 1$, the other cases can be handled similarly. We divide this proof into two cases.

Case 1. $\alpha_1 \leq \alpha_2$.

Firstly, we assume $m \in W^{\alpha_2}(\mathcal{M}_{\mathcal{F}}(M_1, M_2), \mathcal{M}_p(l_{q_1}^{s_1}, l_{q_2}^{s_2}))$. By the definition, we have that $\square_k^{\alpha_2} T_m \in \mathcal{L}(M_1, M_2)$ for $k \in \mathbb{Z}^n$, and $\{\|\square_k^{\alpha_2} T_m\|_{M_1 \rightarrow M_2}\} \in \mathcal{M}_p(l_{q_1}^{s_1, \alpha_2}, l_{q_2}^{s_2, \alpha_2})$. For any $f \in \mathcal{S}$, we deduce

$$\begin{aligned} \|\square_k^{\alpha_2} T_m f\|_{L^{p_2}} &\sim \|\square_k^{\alpha_2} T_m f\|_{M_2} \\ &= \|\square_k^{\alpha_2} T_m \square_k^{\alpha_2, *} f\|_{M_2} \\ &\lesssim \|\square_k^{\alpha_2} T_m\|_{M_1 \rightarrow M_2} \|\square_k^{\alpha_2, *} f\|_{M_1}. \end{aligned} \quad (3.14)$$

It yields

$$\begin{aligned} \|T_m f\|_{M_2^{s_2}} &= \|\{\|\square_k^{\alpha_2} T_m f\|_{L^{p_2}}\}\|_{l_{q_2}^{s_2, \alpha_2}} \\ &\lesssim \|\{\|\square_k^{\alpha_2} T_m\|_{M_1 \rightarrow M_2} \|\square_k^{\alpha_2, *} f\|_{M_1}\}\|_{l_{q_2}^{s_2, \alpha_2}} \\ &\lesssim \|\{\|\square_k^{\alpha_2} T_m\|_{M_1 \rightarrow M_2}\}\|_{l_{q_1}^{s_1, \alpha_2} \rightarrow l_{q_2}^{s_2, \alpha_2}} \|\{\|\square_k^{\alpha_2, *} f\|_{M_1}\}\|_{l_{q_1}^{s_1, \alpha_2}} \\ &= \|m\|_{W^{\alpha_2}(\mathcal{M}_{\mathcal{F}}(M_1, M_2), \mathcal{M}_p(l_{q_1}^{s_1}, l_{q_2}^{s_2}))} \|\{\|\square_k^{\alpha_2, *} f\|_{M_1}\}\|_{l_{q_1}^{s_1, \alpha_2}}. \end{aligned}$$

Observing

$$\begin{aligned} \|\{\|\square_k^{\alpha_2, *} f\|_{M_1}\}\|_{l_{q_1}^{s_1, \alpha_2}} &\lesssim \|\{\sum_{l \in \Lambda_k^{\alpha_2, *}} \|\square_l^{\alpha_2} f\|_{M_1}\}_{k \in \mathbb{Z}^n}\|_{l_{q_1}^{s_1, \alpha_2}} \\ &\lesssim \|\{\|\square_k^{\alpha_2} f\|_{M_1}\}_{k \in \mathbb{Z}^n}\|_{l_{q_1}^{s_1, \alpha_2}} \sim \|f\|_{M_1^{s_1}}, \end{aligned} \quad (3.15)$$

we obtain

$$\|T_m f\|_{M_2^{s_2}} \lesssim \|m\|_{W^{\alpha_2}(\mathcal{M}_{\mathcal{F}}(M_1, M_2), \mathcal{M}_p(l_{q_1}^{s_1}, l_{q_2}^{s_2}))} \|f\|_{M_1^{s_1}},$$

which further implies

$$\|m\|_{\mathcal{M}_{\mathcal{F}}(M_1^{s_1}, M_2^{s_2})} \lesssim \|m\|_{W^{\alpha_2}(\mathcal{M}_{\mathcal{F}}(M_1, M_2), \mathcal{M}_p(l_{q_1}^{s_1}, l_{q_2}^{s_2}))}. \quad (3.16)$$

Next, we assume $m \in \mathcal{M}_{\mathcal{F}}(M_1^{s_1}, M_2^{s_2})$. For the local operator $\square_k^{\alpha_2} T_m$, we have

$$\begin{aligned} \|\square_k^{\alpha_2} T_m f\|_{M_2} &\sim \langle k \rangle^{\frac{-s_2}{1-\alpha_2}} \|T_m(\square_k^{\alpha_2} f)\|_{M_2^{s_2}} \\ &\lesssim \|m\|_{\mathcal{M}_{\mathcal{F}}(M_1^{s_1}, M_2^{s_2})} \|\langle k \rangle^{\frac{-s_2}{1-\alpha_2}} \|\square_k^{\alpha_2} f\|_{M_1^{s_1}} \\ &\lesssim \|m\|_{\mathcal{M}_{\mathcal{F}}(M_1^{s_1}, M_2^{s_2})} \|\langle k \rangle^{\frac{s_1-s_2}{1-\alpha_2}} \|f\|_{M_1}, \end{aligned}$$

which implies $\square_k^{\alpha_2} T_m \in \mathcal{L}(M_1, M_2)$.

Moreover, by the almost disjointization of the decomposition, we choose a subset of \mathbb{Z}^n denoted by E_m , which may depend on the exact m , such that

$$\Lambda_k^{\alpha_2, *} \cap \Lambda_l^{\alpha_2, *} = \emptyset \quad (3.17)$$

for any $k, l \in E_m$, $k \neq l$, and

$$\begin{aligned} &\|m\|_{W^{\alpha_2}(\mathcal{M}_{\mathcal{F}}(M_1, M_2), \mathcal{M}_p(l_{q_1}^{s_1}, l_{q_2}^{s_2}))} \\ &\leq C \|\{\|\square_k^{\alpha_2} T_m\|_{M_1 \rightarrow M_2}\}\|_{l_{q_1}^{s_1, \alpha_2}(E_m) \rightarrow l_{q_2}^{s_2, \alpha_2}(E_m)}, \end{aligned} \quad (3.18)$$

where the constant C is independent of the exact m . We assume without loss of generality that $\|\square_k^{\alpha_2} T_m\|_{M_1 \rightarrow M_2} \neq 0$ for all $k \in E_m$. Then, one can find $f_k \in \mathcal{S}$, $f_k \neq 0$ such that

$$\|\square_k^{\alpha_2} T_m f_k\|_{M_2} \gtrsim \|\square_k^{\alpha_2} T_m\|_{M_1 \rightarrow M_2} \|f_k\|_{M_1} \quad (3.19)$$

for every $k \in E_m$. It leads to

$$\|\square_k^{\alpha_2} T_m \square_k^{\alpha_2, *} f_k\|_{M_2} \gtrsim \|\square_k^{\alpha_2} T_m\|_{M_1 \rightarrow M_2} \|\square_k^{\alpha_2, *} f_k\|_{M_1}. \quad (3.20)$$

In addition, we deduce $\|\square_k^{\alpha_2, *} f_k\|_{M_1} \neq 0$ by the fact $\|\square_k^{\alpha_2} T_m \square_k^{\alpha_2, *} f_k\|_{M_2} \neq 0$.

For any nonnegative sequence $\{a_k\}_{k \in \Gamma_m}$, we have that

$$\begin{aligned} & \|\{a_k \|\square_k^{\alpha_2} T_m\|_{M_1 \rightarrow M_2} \|\square_k^{\alpha_2, *} f_k\|_{M_1}\} | l_{q_2^{s_2, \alpha_2}}(E_m)\| \\ & \lesssim \|\{a_k \|\square_k^{\alpha_2} T_m \square_k^{\alpha_2, *} f_k\|_{M_2}\} | l_{q_2^{s_2, \alpha_2}}(E_m)\| \\ & \lesssim \left\| \sum_{k \in E_m} a_k \|\square_k^{\alpha_2, *} T_m f_k\|_{M_2^{s_2}} \right\| = \|T_m(\sum_{k \in E_m} a_k \square_k^{\alpha_2, *} f_k)\|_{M_2^{s_2}} \\ & \lesssim \|m\| \mathcal{M}_{\mathcal{F}}(M_1^{s_1}, M_2^{s_2}) \left\| \sum_{k \in E_m} a_k \square_k^{\alpha_2, *} f_k \right\|_{M_1^{s_1}}. \end{aligned} \quad (3.21)$$

By the fact that $|\{k \in \mathbb{Z}^n : \square_l^{\alpha_1} \square_k^{\alpha_2, *} \neq 0\}| \leq C$ for any $l \in \mathbb{Z}^n$, we deduce

$$\begin{aligned} \left\| \sum_{k \in E_m} a_k \square_k^{\alpha_2, *} f_k \right\|_{M_1^{s_1}} &= \left(\sum_{l \in \mathbb{Z}^n} \langle l \rangle^{\frac{s_1 q_1}{1-\alpha_1}} \left\| \sum_{k \in E_m} a_k \square_l^{\alpha_1} \square_k^{\alpha_2, *} f_k \right\|_{L^{p_1}^{q_1}}^{q_1} \right)^{1/q_1} \\ &\lesssim \left(\sum_{l \in \mathbb{Z}^n} \langle l \rangle^{\frac{s_1 q_1}{1-\alpha_1}} \sum_{k \in E_m} \|a_k \square_l^{\alpha_1} \square_k^{\alpha_2, *} f_k\|_{L^{p_1}^{q_1}}^{q_1} \right)^{1/q_1} \\ &= \left(\sum_{k \in E_m} \langle k \rangle^{\frac{s_1 q_1}{1-\alpha_2}} a_k^{q_1} \sum_{l \in \mathbb{Z}^n} \|\square_l^{\alpha_1} \square_k^{\alpha_2, *} f_k\|_{L^{p_1}^{q_1}}^{q_1} \right)^{1/q_1} \\ &\sim \|\{a_k \|\square_k^{\alpha_2, *} f_k\|_{M_1}\} | l_{q_1^{s_1, \alpha_2}}(E_m)\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} & \|\{a_k \|\square_k^{\alpha_2} T_m\|_{M_1 \rightarrow M_2} \|\square_k^{\alpha_2, *} f_k\|_{M_1}\} | l_{q_2^{s_2, \alpha_2}}(E_m)\| \\ & \lesssim \|m\| \mathcal{M}_{\mathcal{F}}(M_1^{s_1}, M_2^{s_2}) \|\{a_k \|\square_k^{\alpha_2, *} f_k\|_{M_1}\} | l_{q_1^{s_1, \alpha_2}}(E_m)\|. \end{aligned} \quad (3.22)$$

By the arbitrariness of $\{a_k\}_{k \in E_m}$ and the fact $\|\square_k^{\alpha_2, *} f_k\|_{M_1} \neq 0$, we have

$$\|\{\|\square_k^{\alpha_2} T_m\|_{M_1 \rightarrow M_2}\} | l_{q_1^{s_1, \alpha_2}}(E_m) \rightarrow l_{q_2^{s_2, \alpha_2}}(E_m)\| \lesssim \|m\| \mathcal{M}_{\mathcal{F}}(M_1^{s_1}, M_2^{s_2}). \quad (3.23)$$

So we deduce

$$\|m\| W^{\alpha_2}(\mathcal{M}_{\mathcal{F}}(M_1, M_2), \mathcal{M}_p(l_{q_1^{s_1}}^{s_1}, l_{q_2^{s_2}}^{s_2})) \lesssim \|m\| \mathcal{M}_{\mathcal{F}}(M_1^{s_1}, M_2^{s_2}). \quad (3.24)$$

Case 2. $\alpha_1 > \alpha_2$.

We assume $m \in W^{\alpha_1}(\mathcal{M}_{\mathcal{F}}(M_1, M_2), \mathcal{M}_p(l_{q_1^{s_1}}^{s_1}, l_{q_2^{s_2}}^{s_2}))$. By the definition, we obtain $\square_k^{\alpha_1} T_m \in \mathcal{L}(M_1, M_2)$ for all $k \in \mathbb{Z}^n$, and $\{\|\square_k^{\alpha_1} T_m\|_{M_1 \rightarrow M_2}\} \in \mathcal{M}_p(l_{q_1^{s_1, \alpha_1}}^{s_1, \alpha_1}, l_{q_2^{s_2, \alpha_1}}^{s_2, \alpha_1})$. For any $f \in \mathcal{S}$, we deduce

$$\begin{aligned} \|\square_k^{\alpha_1} T_m f\|_{M_2} &= \|\square_k^{\alpha_1} T_m \square_k^{\alpha_1, *} f\|_{M_2} \\ &\lesssim \|\square_k^{\alpha_1} T_m\|_{M_1 \rightarrow M_2} \|\square_k^{\alpha_1, *} f\|_{M_1}. \end{aligned} \quad (3.25)$$

It yields

$$\begin{aligned}
\|T_m f\|_{M_2^{s_2}} &\sim \|\{\|\square_k^{\alpha_1} T_m f\|_{M_2}\}\|_{l_{q_2}^{s_2, \alpha_1}} \\
&\lesssim \|\{\|\square_k^{\alpha_1} T_m\|_{M_1 \rightarrow M_2} \|\square_k^{\alpha_1, *}\|_{M_1}\}\|_{l_{q_2}^{s_2, \alpha_1}} \\
&\lesssim \|\{\|\square_k^{\alpha_1} T_m\|_{M_1 \rightarrow M_2}\}\|_{l_{q_1}^{s_1, \alpha_1} \rightarrow l_{q_2}^{s_2, \alpha_1}} \|\{\|\square_k^{\alpha_1, *}\|_{M_1}\}\|_{l_{q_1}^{s_1, \alpha_1}} \\
&\lesssim \|m\| W^{\alpha_1}(\mathcal{M}_{\mathcal{F}}(M_1, M_2), \mathcal{M}_p(l_{q_1}^{s_1}, l_{q_2}^{s_2})) \|\|f\|_{M_1^{s_1}},
\end{aligned}$$

which further implies

$$\|m\| \mathcal{M}_{\mathcal{F}}(M_1^{s_1}, M_2^{s_2}) \lesssim \|m\| W^{\alpha_1}(\mathcal{M}_{\mathcal{F}}(M_1, M_2), \mathcal{M}_p(l_{q_1}^{s_1}, l_{q_2}^{s_2})). \quad (3.26)$$

On the other hand, if $m \in \mathcal{M}_{\mathcal{F}}(M_1^{s_1}, M_2^{s_2})$, we deduce $\square_k^{\alpha_1} T_m \in \mathcal{L}(M_1, M_2)$ as in Case 1. By the almost disjointization of the decomposition, we choose a subset of \mathbb{Z}^n denoted by E_m , which may depend on the exact m , such that

$$\Lambda_k^{\alpha_1, *} \cap \Lambda_l^{\alpha_1, *} = \emptyset \quad (3.27)$$

for any $k, l \in E_m$, $k \neq l$, and

$$\begin{aligned}
&\|m\| W^{\alpha_1}(\mathcal{M}_{\mathcal{F}}(M_1, M_2), \mathcal{M}_p(l_{q_1}^{s_1}, l_{q_2}^{s_2})) \\
&\leq C \|\{\|\square_k^{\alpha_1} T_m\|_{M_1 \rightarrow M_2}\}\|_{l_{q_1}^{s_1, \alpha_1}(E_m) \rightarrow l_{q_2}^{s_2, \alpha_1}(E_m)},
\end{aligned} \quad (3.28)$$

where the constant C is independent of the exact m . We assume without loss of generality that $\|\square_k^{\alpha_1} T_m\|_{M_1 \rightarrow M_2} \neq 0$ for all $k \in E_m$. Then, one can find $f_k \in \mathcal{S}$, $f_k \neq 0$ such that

$$\|\square_k^{\alpha_1} T_m f_k\|_{M_2} \gtrsim \|\square_k^{\alpha_1} T_m\|_{M_1 \rightarrow M_2} \|f_k\|_{M_1} \quad (3.29)$$

for every $k \in E_m$, which leads to

$$\|\square_k^{\alpha_1} T_m \square_k^{\alpha_1, *} f_k\|_{M_2} \gtrsim \|\square_k^{\alpha_1} T_m\|_{M_1 \rightarrow M_2} \|\square_k^{\alpha_1, *} f_k\|_{M_1}. \quad (3.30)$$

In addition, we deduce $\|\square_k^{\alpha_1, *} f_k\|_{M_1} \neq 0$ by the fact $\|\square_k^{\alpha_1} T_m \square_k^{\alpha_1, *} f_k\|_{M_2} \neq 0$.

For any nonnegative sequence $\{a_k\}_{k \in \Gamma_m}$, we have that

$$\begin{aligned}
&\|\{a_k \|\square_k^{\alpha_1} T_m\|_{M_1 \rightarrow M_2} \|\square_k^{\alpha_1, *} f_k\|_{M_1}\}\|_{l_{q_2}^{s_2, \alpha_1}(E_m)} \\
&\lesssim \|\{a_k \|\square_k^{\alpha_1} T_m \square_k^{\alpha_1, *} f_k\|_{M_2}\}\|_{l_{q_2}^{s_2, \alpha_1}(E_m)}.
\end{aligned} \quad (3.31)$$

By the spirit of Proposition 3.1, we deduce

$$\begin{aligned}
\|\{a_k \|\square_k^{\alpha_1} T_m \square_k^{\alpha_1, *} f_k\|_{M_2}\}\|_{l_{q_2}^{s_2, \alpha_1}(E_m)} &\lesssim \|\{\|\square_k^{\alpha_1} \sum_{l \in E_m} a_l \square_l^{\alpha_1, *} T_m f_l\|_{M_2}\}\|_{l_{q_2}^{s_2, \alpha_1}} \\
&\sim \|\sum_{k \in E_m} a_k \square_k^{\alpha_1, *} T_m f_k\|_{M_2^{s_2}} \\
&= \|T_m(\sum_{k \in E_m} a_k \square_k^{\alpha_1, *} f_k)\|_{M_2^{s_2}}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\|\{a_k \|\square_k^{\alpha_1} T_m\|_{M_1 \rightarrow M_2} \|\square_k^{\alpha_1, *} f_k\|_{M_1}\}\|_{l_{q_2}^{s_2, \alpha_1}(E_m)} \\
&\lesssim \|T_m(\sum_{k \in E_m} a_k \square_k^{\alpha_1, *} f_k)\|_{M_2^{s_2}} \\
&\lesssim \|m\| \mathcal{M}_{\mathcal{F}}(M_1^{s_1}, M_2^{s_2}) \|\sum_{k \in E_m} a_k \square_k^{\alpha_1, *} f_k\|_{M_1^{s_1}} \\
&\lesssim \|m\| \mathcal{M}_{\mathcal{F}}(M_1^{s_1}, M_2^{s_2}) \|\{a_k \|\square_k^{\alpha_1, *} f_k\|_{M_1}\}\|_{l_{q_1}^{s_1, \alpha_1}(E_m)}.
\end{aligned} \quad (3.32)$$

By the arbitrariness of $\{a_k\}_{k \in E_m}$ and the fact $\|\square_k^{\alpha_1, *}\|_{M_1} \neq 0$, we have

$$\left\| \left\{ \|\square_k^{\alpha_1} T_m\|_{M_1 \rightarrow M_2} \right\} \right\|_{l_{q_1}^{s_1, \alpha_1}(E_m) \rightarrow l_{q_2}^{s_2, \alpha_1}(E_m)} \lesssim \|m\| \mathcal{M}_{\mathcal{F}}(M_1^{s_1}, M_2^{s_2}). \quad (3.33)$$

So we deduce

$$\|m\| W^{\alpha_1}(\mathcal{M}_{\mathcal{F}}(M_1, M_2), \mathcal{M}_p(l_{q_1}^{s_1}, l_{q_2}^{s_2})) \lesssim \|m\| \mathcal{M}_{\mathcal{F}}(M_1^{s_1}, M_2^{s_2}). \quad (3.34)$$

□

It is obvious that the embedding relations between α -modulation spaces can be viewed as the boundedness of the identity operator between the same α -modulation spaces. Using Theorem 3.2, we obtain the reduction of embedding immediately.

Corollary 3.3 (Reduction of the embedding). *Let $0 < p_i, q_i \leq \infty$, $s_i \in \mathbb{R}$, $\alpha_i \in [0, 1]$ for $i = 1, 2$. Then*

$$M_{p_1, q_1}^{s_1, \alpha_1} \subseteq M_{p_2, q_2}^{s_2, \alpha_2} \quad (3.35)$$

holds if and only if

$$\left\{ \left\| \square_k^{\alpha_1 \vee \alpha_2} \right\| M_{p_1, q_1}^{0, \alpha_1} \rightarrow M_{p_2, q_2}^{0, \alpha_2} \right\} \in \mathcal{M}_p(l_{q_1}^{s_1, \alpha_1 \vee \alpha_2}, l_{q_2}^{s_2, \alpha_1 \vee \alpha_2}) \quad (3.36)$$

for $\alpha_1 \vee \alpha_2 < 1$, and

$$\left\{ \left\| \Delta_j \right\| M_{p_1, q_1}^{0, \alpha_1} \rightarrow M_{p_2, q_2}^{0, \alpha_2} \right\} \in \mathcal{M}_p(l_{q_1}^{s_1, 1}, l_{q_2}^{s_2, 1}) \quad (3.37)$$

for $\alpha_1 \vee \alpha_2 = 1$.

4. ASYMPTOTIC ESTIMATES FOR LOCAL OPERATORS

In this section, we establish some asymptotic estimates for local operators between α -modulation spaces. For $0 < p_1, p_2, q \leq \infty$ and $(\alpha_1, \alpha_2) \in [0, 1] \times [0, 1]$, we denote

$$A(\mathbf{p}, q; \alpha_1, \alpha_2) = \begin{cases} \left[n\alpha_1 \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \right] \vee \left[n\alpha_2 \left(1 - \frac{1}{p_2} \right) - n\alpha_1 \left(1 - \frac{1}{p_1} \right) - n(\alpha_2 - \alpha_1) \frac{1}{q} \right] \\ \quad \vee \left[n(\alpha_2 - \alpha_1) \left(\frac{1}{p_2} - \frac{1}{q} \right) + n\alpha_1 \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \right], \text{ if } \alpha_1 \leq \alpha_2, \\ \left[n\alpha_2 \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \right] \vee \left[n\alpha_2 \left(1 - \frac{1}{p_2} \right) - n\alpha_1 \left(1 - \frac{1}{p_1} \right) - n(\alpha_2 - \alpha_1) \frac{1}{q} \right] \\ \quad \vee \left[n(\alpha_1 - \alpha_2) \left(\frac{1}{q} - \frac{1}{p_1} \right) + n\alpha_2 \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \right], \text{ if } \alpha_1 > \alpha_2. \end{cases}$$

Lemma 4.1 (Asymptotic estimates). *Let $0 < p_1 \leq p_2 \leq \infty$, $0 < q \leq \infty$, $s_i \in \mathbb{R}$, $\alpha_i \in [0, 1]$ for $i = 1, 2$. We have*

$$\left\| \square_k^{\alpha_1 \vee \alpha_2} \right\| M_{p_1, q}^{0, \alpha_1} \rightarrow M_{p_2, q}^{0, \alpha_2} \sim \langle k \rangle^{\frac{A(\mathbf{p}, q, \alpha_1, \alpha_2)}{1 - \alpha_1 \vee \alpha_2}} \quad (4.1)$$

for $\alpha_1 \vee \alpha_2 < 1$ and $k \in \mathbb{Z}^n$. Also

$$\left\| \Delta_j \right\| M_{p_1, q}^{0, \alpha_1} \rightarrow M_{p_2, q}^{0, \alpha_2} \sim 2^{jA(\mathbf{p}, q, \alpha_1, \alpha_2)} \quad (4.2)$$

for $\alpha_1 \vee \alpha_2 = 1$ and $j \in \{0\} \cup \mathbb{Z}^+$.

Proof. In this proof, we denote $M_i = M_{p_i, q}^{0, \alpha_i}$ for simplicity. We only state the proof for the case $\alpha_1, \alpha_2 < 1$, since the proof of other cases are similar.

Case 1. $\alpha_1 \leq \alpha_2 < 1$. In this case, we need to show

$$\|\square_k^{\alpha_2} | M_1 \rightarrow M_2\| \sim \langle k \rangle^{\frac{A(\mathbf{p}, q, \alpha_1, \alpha_2)}{1-\alpha_2}} \sim 2^{jA(\mathbf{p}, q, \alpha_1, \alpha_2)}. \quad (4.3)$$

for each $k \in \mathbb{Z}^n$, $j \in \{0\} \cup \mathbb{Z}^+$ and $\langle k \rangle^{\frac{1}{1-\alpha_2}} \sim 2^j$.

Denote

$$\begin{aligned} A_1(\mathbf{p}, q, \alpha_1, \alpha_2) &= n\alpha_1(1/p_1 - 1/p_2), \\ A_2(\mathbf{p}, q, \alpha_1, \alpha_2) &= n\alpha_2(1 - 1/p_2) - n\alpha_1(1 - 1/p_1) - n(\alpha_2 - \alpha_1)/q, \\ A_3(\mathbf{p}, q, \alpha_1, \alpha_2) &= n(\alpha_2 - \alpha_1)(1/p_2 - 1/q) + n\alpha_1(1/p_1 - 1/p_2). \end{aligned} \quad (4.4)$$

Obviously, we have $A(\mathbf{p}, q, \alpha_1, \alpha_2) = \max_{i=1,2,3} A_i(\mathbf{p}, q, \alpha_1, \alpha_2)$ for $\alpha_1 \leq \alpha_2$.

Lower bound estimates. In this part, we use some special functions to test the operator norms. Take a smooth function f whose Fourier transform \widehat{f} has small support near the origin such that $\text{supp } \widehat{f}_k^\alpha \subset \widetilde{\text{supp } \eta_k^\alpha}$ for every $k \in \mathbb{Z}^n$, $\alpha \in [0, 1)$, where we denote

$$\widehat{f}_k^\alpha = \widehat{f}\left(\frac{\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k}{\langle k \rangle^{\frac{1}{1-\alpha}}}\right). \quad (4.5)$$

Firstly, we have

$$\begin{aligned} \|\square_k^{\alpha_2} | M_1 \rightarrow M_2\| &\gtrsim \frac{\|\square_k^{\alpha_2} f_l^{\alpha_1}\|_{M_2}}{\|f_l^{\alpha_1}\|_{M_1}} \sim \frac{\|f_l^{\alpha_1}\|_{L^{p_2}}}{\|f_l^{\alpha_1}\|_{L^{p_1}}} \sim 2^{jn\alpha_1(1/p_1 - 1/p_2)} \\ &= 2^{jA_1(\mathbf{p}, q, \alpha_1, \alpha_2)} \end{aligned} \quad (4.6)$$

for some suitable $l \in \mathbb{Z}^n$ such that $\langle l \rangle^{\frac{1}{1-\alpha_1}} \sim \langle k \rangle^{\frac{1}{1-\alpha_2}} \sim 2^j$.

Next, a direct calculation yields that

$$\|\square_k^{\alpha_2} f_k^{\alpha_2}\|_{M_2} = \|f_k^{\alpha_2}\|_{M_2} \sim \|f_k^{\alpha_2}\|_{L^{p_2}} \sim 2^{jn\alpha_2(1-1/p_2)} \quad (4.7)$$

and

$$\begin{aligned} \|f_k^{\alpha_2}\|_{M_1} &= \left(\sum_{l \in \mathbb{Z}^n} \langle l \rangle^{\frac{sq}{1-\alpha_1}} \|\square_l^{\alpha_1} f_k^{\alpha_2}\|_{L^p}^q \right)^{1/q} \lesssim \left(\sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \langle l \rangle^{\frac{sq}{1-\alpha_1}} \|\mathcal{F}^{-1} \eta_l^{\alpha_1}\|_{L^p}^q \right)^{\frac{1}{q}} \\ &\lesssim 2^{j\alpha_1 n(1-1/p_1)} 2^{j(\alpha_2 - \alpha_1)n/q}. \end{aligned} \quad (4.8)$$

So we have

$$\begin{aligned} \|\square_k^{\alpha_2} | M_1 \rightarrow M_2\| &\gtrsim \frac{\|\square_k^{\alpha_2} f_k^{\alpha_2}\|_{M_2}}{\|f_k^{\alpha_2}\|_{M_1}} \gtrsim \frac{2^{jn\alpha_2(1-1/p_2)}}{2^{j\alpha_1 n(1-1/p_1)} 2^{j(\alpha_2 - \alpha_1)n/q}} \\ &= 2^{jn\alpha_2(1-1/p_2)} 2^{-jn\alpha_1(1-1/p_1)} 2^{-jn(\alpha_2 - \alpha_1)/q} \\ &= 2^{jA_2(\mathbf{p}, q, \alpha_1, \alpha_2)}. \end{aligned} \quad (4.9)$$

Finally, let

$$F_{k,N} = \sum_{l \in \widetilde{\Gamma_k^{\alpha_1, \alpha_2}}} T_{Nl} f_l^{\alpha_1}, \quad (4.10)$$

where T_{Nl} denotes the translation operator: $T_{Nl}f(x) = f(x - Nl)$. By the almost orthogonality of $\{T_{Nl}f_l^{\alpha_1}\}_{l \in \widetilde{\Gamma_k^{\alpha_1, \alpha_2}}}$ as $N \rightarrow \infty$, we deduce that

$$\|\square_k^{\alpha_2} F_{k,N}\|_{M_2} = \|F_{k,N}\|_{M_2} = \|F_{k,N}\|_{L^{p_2}} \sim 2^{jn(\alpha_2 - \alpha_1)/p_2} 2^{jn\alpha_1(1-1/p_2)} \quad (4.11)$$

as $N \rightarrow \infty$.

On the other hand,

$$\begin{aligned} \|F_{k,N}\|_{M_1} &= \left(\sum_{l \in \widetilde{\Gamma_k^{\alpha_1, \alpha_2}}} \|f_l^{\alpha_1}\|_{L^{p_1}}^q \right)^{\frac{1}{q}} \\ &\sim |\widetilde{\Gamma_k^{\alpha_1, \alpha_2}}|^{1/q} 2^{jn\alpha_1(1-1/p_1)} \\ &\sim 2^{jn(\alpha_2-\alpha_1)/q} 2^{jn\alpha_1(1-1/p_1)}. \end{aligned} \quad (4.12)$$

So by the definition of operator norm, we have

$$\begin{aligned} \|\square_k^{\alpha_2} | M_1 \rightarrow M_2\| &\gtrsim \lim_{N \rightarrow \infty} \frac{\|\square_k^{\alpha_2} F_{k,N}\|_{M_2}}{\|F_{k,N}\|_{M_1}} \sim \frac{2^{jn(\alpha_2-\alpha_1)/p_2} 2^{jn\alpha_1(1-1/p_2)}}{2^{jn(\alpha_2-\alpha_1)/q} 2^{jn\alpha_1(1-1/p_1)}} \\ &\sim 2^{jn(\alpha_2-\alpha_1)(1/p_2-1/q)} 2^{jn\alpha_1(1/p_1-1/p_2)} \\ &= 2^{jA_3(\mathbf{p}, q, \alpha_1, \alpha_2)}. \end{aligned} \quad (4.13)$$

Taking together these estimates, we have the lower bounds

$$\|\square_k^{\alpha_2} | M_1 \rightarrow M_2\| \gtrsim \langle k \rangle^{\frac{A_i(\mathbf{p}, q, \alpha_1, \alpha_2)}{1-\alpha}} \quad (4.14)$$

for $i = 1, 2, 3$. Recalling $A(\mathbf{p}, q, \alpha_1, \alpha_2) = \max_{i=1,2,3} A_i(\mathbf{p}, q, \alpha_1, \alpha_2)$, we complete the lower bound estimates.

Upper bound estimates. Now, we turn to the estimate of upper bound. Denote

$$\begin{aligned} S &= \{(1/p_1, 1/p_2, 1/q) \in [0, \infty)^3 : 1/p_2 \leq 1/p_1\}, \\ S_1 &= S \cap \{(1/p_1, 1/p_2, 1/q) : 1/q \geq 1 - 1/p_2, 1/q \geq 1/p_2\}, \\ S_2 &= S \cap \{(1/p_1, 1/p_2, 1/q) : 1/q \leq 1 - 1/p_2, 1/p_2 \leq 1/2\}, \\ S_3 &= S \cap \{(1/p_1, 1/p_2, 1/q) : 1/q \leq 1/p_2, 1/p_2 \geq 1/2\}. \end{aligned}$$

Obviously, we have

$$S = S_1 \cup S_2 \cup S_3,$$

and

$$A(\mathbf{p}, q, \alpha_1, \alpha_2) = A_i(\mathbf{p}, q, \alpha_1, \alpha_2)$$

for $(1/p_1, 1/p_2, 1/q) \in S_i, \alpha_1 \leq \alpha_2$. To verify $\|\square_k^{\alpha_2} | M_1 \rightarrow M_2\| \lesssim \langle k \rangle^{\frac{A(\mathbf{p}, q, \alpha_1, \alpha_2)}{1-\alpha_2}}$, we only need to verify that

$$\|\square_k^{\alpha_2} | M_1 \rightarrow M_2\| \lesssim \langle k \rangle^{\frac{A_j(\mathbf{p}, q, \alpha_1, \alpha_2)}{1-\alpha_2}}$$

in S_j for $j = 1, 2, 3$.

For S_1 , we want to verify

$$\|\square_k^{\alpha_2} | M_1 \rightarrow M_2\| \lesssim \langle k \rangle^{\frac{A_1(\mathbf{p}, q, \alpha_1, \alpha_2)}{1-\alpha_2}}. \quad (4.15)$$

In fact, once we obtain the estimates for the following 4 cases, the upper bound in S_1 can be deduced by Lemma 2.4.

Case 1.1. $1/p_2 = 1/q = 1/2$, $1/p_2 \leq 1/p_1$. We have

$$\|\square_k^{\alpha_2} f\|_{M_2} = \|\square_k^{\alpha_2} f\|_{M_{2,2}^{0,\alpha_2}} = \|\square_k^{\alpha_2} f\|_{M_{2,2}^{0,\alpha_1}} = \left(\sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} \square_k^{\alpha_2} f\|_{L^2}^2 \right)^{1/2}. \quad (4.16)$$

Then we use Lemma 2.2 and Lemma 2.3 to deduce that

$$\begin{aligned} \|\square_k^{\alpha_2} f\|_{M_2} &\lesssim \left(\sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^2}^2 \right)^{1/2} \\ &\lesssim 2^{jn\alpha_1(1/p_1-1/p_2)} \left(\sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^{p_1}}^2 \right)^{1/2} \\ &\lesssim 2^{jn\alpha_1(1/p_1-1/p_2)} \|f\|_{M_1}. \end{aligned} \quad (4.17)$$

Moreover, in this case, we have

$$2^{jn\alpha_1(1/p_1-1/p_2)} = 2^{jA_1(\mathbf{p}, q, \alpha_1, \alpha_2)} = 2^{jA_2(\mathbf{p}, q, \alpha_1, \alpha_2)} = 2^{jA_3(\mathbf{p}, q, \alpha_1, \alpha_2)}. \quad (4.18)$$

Case 1.2. $1/p_2 = 1/p_1 = 0$, $1/q \geq 1$. We have

$$\begin{aligned} \|\square_k^{\alpha_2} f\|_{M_2} &= \|\square_k^{\alpha_2} f\|_{L^\infty} \lesssim \sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^\infty} \\ &\lesssim \left(\sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^\infty}^q \right)^{1/q} \lesssim \|f\|_{M_1} = 2^{jA_1(\mathbf{p}, q, \alpha_1, \alpha_2)} \|f\|_{M_1}. \end{aligned} \quad (4.19)$$

Case 1.3. $1/p_2 = 1/p_1 = 1/q \geq 1$. We use Lemma 2.3 to deduce that

$$\begin{aligned} \|\square_k^{\alpha_2} f\|_{M_2} &= \|\square_k^{\alpha_2} f\|_{L^{p_2}} = \left\| \sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \square_l^{\alpha_1} \square_k^{\alpha_2} f \right\|_{L^{p_2}} \\ &\lesssim \left(\sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^{p_2}}^{p_2} \right)^{1/p_2} \lesssim \|f\|_{M_1}. \end{aligned} \quad (4.20)$$

Moreover, in this case, we have

$$1 = 2^{jA_1(\mathbf{p}, q, \alpha_1, \alpha_2)} = 2^{jA_3(\mathbf{p}, q, \alpha_1, \alpha_2)}. \quad (4.21)$$

Case 1.4. $1/p_2 = 0$, $1/q = 1$. We have that

$$\begin{aligned} \|\square_k^{\alpha_2} f\|_{M_2} &= \|\square_k^{\alpha_2} f\|_{L^\infty} \lesssim \sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^\infty} \\ &\lesssim 2^{jn\alpha_1(1/p_1-1/p_2)} \sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^{p_1}} \\ &\lesssim 2^{jn\alpha_1(1/p_1-1/p_2)} \|f\|_{M_1}. \end{aligned} \quad (4.22)$$

Moreover, in this case, we have

$$2^{jn\alpha_1(1/p_1-1/p_2)} = 2^{jA_1(\mathbf{p}, q, \alpha_1, \alpha_2)} = 2^{jA_2(\mathbf{p}, q, \alpha_1, \alpha_2)}. \quad (4.23)$$

Combining with the estimates of Case 1.1, Case 1.2, Case 1.3, Case 1.4, we use the interpolation theory to obtain the upper bound estimates for S_1 .

For S_2 , we want to verify

$$\|\square_k^{\alpha_2} | M_1 \rightarrow M_2\| \lesssim \langle k \rangle^{\frac{A_2(\mathbf{p}, q, \alpha_1, \alpha_2)}{1-\alpha_2}}. \quad (4.24)$$

To this end, we need the estimates in the following Case 1.5 and Case 1.6.

Case 1.5. $1/2 = 1/p_2 \leq 1/p_1$, $1/q = 0$. We have that

$$\|\square_k^{\alpha_2} f\|_{M_2} = \|\square_k^{\alpha_2} f\|_{L^2} = \|\square_k^{\alpha_2} f\|_{M_{2,2}^{0,\alpha_1}}. \quad (4.25)$$

It then yields

$$\begin{aligned} \|\square_k^{\alpha_2} f\|_{M_2} &\lesssim \left(\sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^2}^2 \right)^{1/2} \\ &\lesssim |\Gamma_k^{\alpha_1, \alpha_2}|^{1/2} \sup_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^2} \\ &\lesssim |\Gamma_k^{\alpha_1, \alpha_2}|^{1/2} 2^{jn\alpha_1(1/p_1-1/p_2)} \sup_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^{p_1}} \\ &\lesssim 2^{jn(\alpha_2-\alpha_1)/2} 2^{jn\alpha_1(1/p_1-1/p_2)} \|f\|_{M_{p_1,q}^{0,\alpha_1}} \sim 2^{jA_2(\mathbf{p}, q, \alpha_1, \alpha_2)} \|f\|_{M_1}. \end{aligned} \quad (4.26)$$

Moreover, in this case, we have

$$2^{jA_2(\mathbf{p}, q, \alpha_1, \alpha_2)} = 2^{jA_3(\mathbf{p}, q, \alpha_1, \alpha_2)}. \quad (4.27)$$

Case 1.6. $1/p_2 = 1/q = 0$. We have

$$\|\square_k^{\alpha_2} f\|_{M_2} = \|\square_k^{\alpha_2} f\|_{L^\infty} \lesssim \sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^\infty}. \quad (4.28)$$

It leads to

$$\begin{aligned} \|\square_k^{\alpha_2} f\|_{M_2} &\lesssim \sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^\infty} \\ &\lesssim |\Gamma_k^{\alpha_1, \alpha_2}| \sup_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^\infty} \\ &\lesssim |\Gamma_k^{\alpha_1, \alpha_2}| 2^{jn\alpha_1(1/p_1-1/p_2)} \sup_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^{p_1}} \\ &\lesssim 2^{jn(\alpha_2-\alpha_1)} 2^{jn\alpha_1(1/p_1-1/p_2)} \|f\|_{M_{p_1,q}^{0,\alpha_1}} \sim 2^{jA_2(\mathbf{p}, q, \alpha_1, \alpha_2)} \|f\|_{M_1}. \end{aligned} \quad (4.29)$$

Combining with the estimates of Case 1.1, Case 1.4, Case 1.5, Case 1.6, we use the interpolation theory to deduce the upper bound estimates for S_2 .

For S_3 , we want to verify

$$\|\square_k^{\alpha_2} | M_1 \rightarrow M_2\| \lesssim \langle k \rangle^{\frac{A_3(\mathbf{p}, q, \alpha_1, \alpha_2)}{1-\alpha_2}}. \quad (4.30)$$

We further need the estimate in following.

Case 1.7. $1/p_2 = 1/p_1 \geq 1$, $1/q = 0$. In this case, we have that

$$\begin{aligned}
\|\square_k^{\alpha_2} f\|_{M_2} &= \|\square_k^{\alpha_2} f\|_{L^{p_2}} = \left\| \sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \square_l^{\alpha_1} \square_k^{\alpha_2} f \right\|_{L^{p_2}} \\
&\lesssim \left(\sum_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^{p_2}}^{p_2} \right)^{1/p_2} \\
&\lesssim |\Gamma_k^{\alpha_1, \alpha_2}|^{1/p_2} \sup_{l \in \Gamma_k^{\alpha_1, \alpha_2}} \|\square_l^{\alpha_1} f\|_{L^{p_2}} \\
&\lesssim 2^{jn(\alpha_2 - \alpha_1)/p_2} \|f\|_{M_1} \sim 2^{jA_3(\mathbf{p}, q, \alpha_1, \alpha_2)} \|f\|_{M_1}.
\end{aligned} \tag{4.31}$$

Combining with the estimates in Case 1.1, Case 1.3, Case 1.5, Case 1.7, we use the interpolation theory to deduce the upper bound estimates for S_3 .

Case 2. $\alpha_2 < \alpha_1 < 1$. In this case, we need to show

$$\|\square_k^{\alpha_1} | M_1 \rightarrow M_2 \| \sim \langle k \rangle^{\frac{A(\mathbf{p}, q, \alpha_1, \alpha_2)}{1 - \alpha_1}} \sim 2^{jA(\mathbf{p}, q, \alpha_1, \alpha_2)}. \tag{4.32}$$

for each $k \in \mathbb{Z}^n$, $j \in \{0\} \cup \mathbb{Z}^+$ and $\langle k \rangle^{\frac{1}{1 - \alpha_1}} \sim 2^j$.

Denote

$$\begin{aligned}
\widetilde{A}_1(\mathbf{p}, q, \alpha_1, \alpha_2) &= n\alpha_2(1/p_1 - 1/p_2) \\
\widetilde{A}_2(\mathbf{p}, q, \alpha_1, \alpha_2) &= n\alpha_2(1 - 1/p_2) - n\alpha_1(1 - 1/p_1) - n(\alpha_2 - \alpha_1)/q \\
\widetilde{A}_3(\mathbf{p}, q, \alpha_1, \alpha_2) &= n(\alpha_1 - \alpha_2)(1/q - 1/p_1) + n\alpha_2(1/p_1 - 1/p_2).
\end{aligned} \tag{4.33}$$

We have $A(\mathbf{p}, q, \alpha_1, \alpha_2) = \max_{i=1,2,3} \widetilde{A}_i(\mathbf{p}, q, \alpha_1, \alpha_2)$ for $\alpha_1 > \alpha_2$.

Lower bound estimates. As in the Case 1, we take a smooth function f whose Fourier transform \widehat{f} has small support near the origin such that $\text{supp } \widehat{f}_k^\alpha \subset \text{supp } \eta_k^\alpha$ for every $k \in \mathbb{Z}^n$, $\alpha \in [0, 1)$.

By the same method in Case 1, we have the following estimates. Firstly, we have

$$\|\square_k^{\alpha_1} | M_1 \rightarrow M_2 \| \gtrsim \frac{\|\square_k^{\alpha_1} f_l^{\alpha_2}\|_{M_2}}{\|f_l^{\alpha_2}\|_{M_1}} \sim 2^{j\widetilde{A}_1(\mathbf{p}, q, \alpha_1, \alpha_2)} \tag{4.34}$$

for some suitable $l \in \mathbb{Z}^n$ such that $\langle l \rangle^{\frac{1}{1 - \alpha_1}} \sim \langle k \rangle^{\frac{1}{1 - \alpha_2}} \sim 2^j$.

We also deduce

$$\|\square_k^{\alpha_1} | M_1 \rightarrow M_2 \| \gtrsim \frac{\|\square_k^{\alpha_1} f_k^{\alpha_1}\|_{M_2}}{\|f_k^{\alpha_1}\|_{M_1}} \gtrsim 2^{j\widetilde{A}_2(\mathbf{p}, q, \alpha_1, \alpha_2)}. \tag{4.35}$$

Finally, take

$$G_{k,N} = \sum_{l \in \widetilde{\Gamma}_k^{\alpha_2, \alpha_1}} T_{Nl} f_l^{\alpha_2} \tag{4.36}$$

We deduce

$$\|\square_k^{\alpha_1} | M_1 \rightarrow M_2 \| \gtrsim \lim_{N \rightarrow \infty} \frac{\|\square_k^{\alpha_1} G_{k,N}\|_{M_2}}{\|G_{k,N}\|_{M_1}} \sim 2^{j\widetilde{A}_3(\mathbf{p}, q, \alpha_1, \alpha_2)}. \tag{4.37}$$

Taking together these estimates, we have the lower bounds

$$\|\square_k^{\alpha_1} | M_1 \rightarrow M_2 \| \gtrsim \langle k \rangle^{\frac{\widetilde{A}_i(\mathbf{p}, q, \alpha_1, \alpha_2)}{1 - \alpha_1}} \tag{4.38}$$

for $i = 1, 2, 3$. Recalling $A(\mathbf{p}, q, \alpha_1, \alpha_2) = \max_{i=1,2,3} \widetilde{A}_i(\mathbf{p}, q, \alpha_1, \alpha_2)$, we complete the lower bound estimates for this case.

Upper bound estimates. We turn to the estimate of upper bound in this case. We divide the area

$$S = \{(1/p_1, 1/p_2, 1/q) \in [0, \infty)^3 : 1/p_2 \leq 1/p_1\}$$

into 3 zones as following.

$$\widetilde{S}_1 = S \cap \{(1/p_1, 1/p_2, 1/q) : 1/q \leq 1 - 1/p_1, 1/q \leq 1/p_1\},$$

$$\widetilde{S}_2 = S \cap \{(1/p_1, 1/p_2, 1/q) : 1/q \geq 1 - 1/p_1, 1/p_1 \geq 1/2\},$$

$$\widetilde{S}_3 = S \cap \{(1/p_1, 1/p_2, 1/q) : 1/q \geq 1/p_1, 1/p_1 \leq 1/2\}.$$

One can easily verify that

$$A(\mathbf{p}, q, \alpha_1, \alpha_2) = \widetilde{A}_i(\mathbf{p}, q, \alpha_1, \alpha_2)$$

for $(1/p_1, 1/p_2, 1/q) \in \widetilde{S}_i$.

For \widetilde{S}_1 , we want to verify

$$\|\square_k^{\alpha_1} | M_1 \rightarrow M_2\| \lesssim \langle k \rangle^{\frac{\widetilde{A}_1(\mathbf{p}, q, \alpha_1, \alpha_2)}{1-\alpha_1}}. \quad (4.39)$$

We deduce the estimates for the following 4 cases, then the upper bound in \widetilde{S}_1 can be deduced by Lemma 2.4.

Case 2.1. $1/p_1 = 1/q = 1/2$, $1/p_2 \leq 1/p_1$. Using Lemma 2.2, We deduce

$$\begin{aligned} \|\square_k^{\alpha_1} f\|_{M_2} &= \left(\sum_{l \in \Gamma_k^{\alpha_2, \alpha_1}} \|\square_l^{\alpha_2} \square_k^{\alpha_1} f\|_{L^{p_2}}^2 \right)^{1/2} \\ &\lesssim 2^{j\alpha_2 n(1/p_1 - 1/p_2)} \left(\sum_{l \in \Gamma_k^{\alpha_2, \alpha_1}} \|\square_l^{\alpha_2} \square_k^{\alpha_1} f\|_{L^2}^2 \right)^{1/2} \\ &= 2^{j\alpha_2 n(1/p_1 - 1/p_2)} \|\square_k^{\alpha_1} f\|_{M_{2,2}^{\alpha_2}} \\ &\sim 2^{j\alpha_2 n(1/p_1 - 1/p_2)} \|\square_k^{\alpha_1} f\|_{M_{2,2}^{\alpha_1}} \lesssim 2^{j\alpha_2 n(1/p_1 - 1/p_2)} \|f\|_{M_1}. \end{aligned} \quad (4.40)$$

Moreover, in this case, we have

$$2^{jn\alpha_2(1/p_1 - 1/p_2)} = 2^{j\widetilde{A}_1(\mathbf{p}, q, \alpha_1, \alpha_2)} = 2^{j\widetilde{A}_2(\mathbf{p}, q, \alpha_1, \alpha_2)} = 2^{j\widetilde{A}_3(\mathbf{p}, q, \alpha_1, \alpha_2)}. \quad (4.41)$$

Case 2.2. $1/p_2 = 1/p_1 = 0$. We obtain

$$\begin{aligned} \|\square_k^{\alpha_1} f\|_{M_2} &= \left(\sum_{l \in \Gamma_k^{\alpha_2, \alpha_1}} \|\square_l^{\alpha_2} \square_k^{\alpha_1} f\|_{L^\infty}^q \right)^{1/q} \\ &\lesssim |\Gamma_k^{\alpha_2, \alpha_1}|^{1/q} \|\square_k^{\alpha_1} f\|_{L^\infty} \\ &\lesssim 2^{j(\alpha_1 - \alpha_2)n/q} \|f\|_{M_1} \sim 2^{j\widetilde{A}_3(\mathbf{p}, q, \alpha_1, \alpha_2)} \|f\|_{M_1}. \end{aligned} \quad (4.42)$$

We also notice that $\widetilde{A}_3(\mathbf{p}, q, \alpha_1, \alpha_2) = \widetilde{A}_1(\mathbf{p}, q, \alpha_1, \alpha_2)$ at $1/p_1 = 1/p_2 = 1/q = 0$.

Case 2.3. $1/p_2 = 1/p_1 \geq 1$, $1/q = 0$. We have

$$\|\square_k^{\alpha_1}\|_{M_2} = \sup_{l \in \Gamma_k^{\alpha_2, \alpha_1}} \|\square_l^{\alpha_2} \square_k^{\alpha_1} f\|_{L^{p_2}}. \quad (4.43)$$

Using Lemma 2.3, we obtain

$$\begin{aligned} \|\square_l^{\alpha_2} \square_k^{\alpha_1} f\|_{L^{p_2}} &\lesssim \langle k \rangle^{\frac{\alpha_1 n(1/p_2-1)}{1-\alpha_1}} \|\mathcal{F}^{-1} \eta_l^{\alpha_2}\|_{L^{p_2}} \|\square_k^{\alpha_1} f\|_{L^{p_2}} \\ &\sim 2^{jn(\alpha_2-\alpha_1)(1-1/p_2)} \|\square_k^{\alpha_1} f\|_{L^{p_2}}. \end{aligned} \quad (4.44)$$

Thus

$$\begin{aligned} \|\square_k^{\alpha_1}\|_{M_2} &\lesssim 2^{jn(\alpha_2-\alpha_1)(1-1/p_2)} \|\square_k^{\alpha_1} f\|_{L^{p_1}} \\ &\lesssim 2^{jn(\alpha_2-\alpha_1)(1-1/p_2)} \|f\|_{M_1} = 2^{j\widetilde{A}_2(\mathbf{p}, q, \alpha_1, \alpha_2)} \|f\|_{M_1}. \end{aligned} \quad (4.45)$$

We also notice that $\widetilde{A}_2(\mathbf{p}, q, \alpha_1, \alpha_2) = \widetilde{A}_1(\mathbf{p}, q, \alpha_1, \alpha_2)$ at $1/p_2 = 1/p_1 = 1$, $1/q = 0$.

Case 2.4. $1/p_2 = 1/q = 0$, $1/p_1 \geq 1$. In this case, we have

$$\begin{aligned} \|\square_k^{\alpha_1}\|_{M_2} &= \sup_{l \in \Gamma_k^{\alpha_2, \alpha_1}} \|\square_l^{\alpha_2} \square_k^{\alpha_1} f\|_{L^\infty} \\ &\lesssim 2^{jn\alpha_2/p_1} \sup_{l \in \Gamma_k^{\alpha_2, \alpha_1}} \|\square_l^{\alpha_2} \square_k^{\alpha_1} f\|_{L^{p_1}} \\ &\lesssim 2^{jn\alpha_2/p_1} 2^{jn(\alpha_2-\alpha_1)(1-1/p_1)} \|\square_k^{\alpha_1} f\|_{L^{p_1}} \\ &\sim 2^{j\widetilde{A}_2(\mathbf{p}, q, \alpha_1, \alpha_2)} \|\square_k^{\alpha_1} f\|_{L^{p_1}} \lesssim 2^{j\widetilde{A}_2(\mathbf{p}, q, \alpha_1, \alpha_2)} \|f\|_{M_1}. \end{aligned} \quad (4.46)$$

We also notice that $\widetilde{A}_2(\mathbf{p}, q, \alpha_1, \alpha_2) = \widetilde{A}_1(\mathbf{p}, q, \alpha_1, \alpha_2)$ at $1/p_2 = 1/q = 0$, $1/p_1 = 1$.

Combining with the estimates of Case 2.1, Case 2.2, Case 2.3, Case 2.4, we use the interpolation theory to obtain the upper bound estimates for \widetilde{S}_1 .

For \widetilde{S}_2 , we want to verify

$$\|\square_k^{\alpha_1} | M_1 \rightarrow M_2 \| \lesssim \langle k \rangle^{\frac{\widetilde{A}_2(\mathbf{p}, q, \alpha_1, \alpha_2)}{1-\alpha_1}}. \quad (4.47)$$

To this end, we need the estimates in the following Case 2.5 and Case 2.6.

Case 2.5. $1/p_2 = 1/p_1 = 1/2$, $1/q \geq 1/2$. By the Hölder's inequality, we deduce

$$\begin{aligned} \|\square_k^{\alpha_1}\|_{M_2} &= \left(\sum_{l \in \Gamma_k^{\alpha_2, \alpha_1}} \|\square_l^{\alpha_2} \square_k^{\alpha_1} f\|_{L^2}^q \right)^{1/q} \\ &\lesssim |\Gamma_k^{\alpha_2, \alpha_1}|^{1/q-1/2} \left(\sum_{l \in \Gamma_k^{\alpha_2, \alpha_1}} \|\square_l^{\alpha_2} \square_k^{\alpha_1} f\|_{L^2}^2 \right)^{1/2} \\ &\sim 2^{jn(\alpha_1-\alpha_2)(1/q-1/2)} \|\square_k^{\alpha_1} f\|_{L^2} \\ &\lesssim 2^{jn(\alpha_1-\alpha_2)(1/q-1/2)} \|f\|_{M_1}. \end{aligned} \quad (4.48)$$

Moreover, we have

$$2^{jn(\alpha_1-\alpha_2)(1/q-1/2)} = 2^{j\widetilde{A}_2(\mathbf{p}, q, \alpha_1, \alpha_2)} = 2^{j\widetilde{A}_3(\mathbf{p}, q, \alpha_1, \alpha_2)} \quad (4.49)$$

in this case.

Case 2.6. $1/p_2 = 0$, $1/p_1 = 1/2$, $1/q \geq 1/2$. Using Lemma 2.2

$$\begin{aligned} \|\square_k^{\alpha_1} f\|_{M_2} &= \left(\sum_{l \in \Gamma_k^{\alpha_2, \alpha_1}} \|\square_l^{\alpha_2} \square_k^{\alpha_1} f\|_{L^\infty}^q \right)^{1/q} \\ &\lesssim 2^{jn\alpha_2/2} \left(\sum_{l \in \Gamma_k^{\alpha_2, \alpha_1}} \|\square_l^{\alpha_2} \square_k^{\alpha_1} f\|_{L^2}^q \right)^{1/q}. \end{aligned} \quad (4.50)$$

Then, we use the conclusion of Case 2.5 to deduce

$$2^{jn\alpha_2/2} \left(\sum_{l \in \Gamma_k^{\alpha_2, \alpha_1}} \|\square_l^{\alpha_2} \square_k^{\alpha_1} f\|_{L^2}^q \right)^{1/q} \lesssim 2^{jn\alpha_2/2} 2^{jn(\alpha_1 - \alpha_2)(1/q - 1/2)} \|f\|_{M_1}. \quad (4.51)$$

It follows that

$$\|\square_k^{\alpha_1} f\|_{M_2} \lesssim 2^{jn\alpha_2/2} 2^{jn(\alpha_1 - \alpha_2)(1/q - 1/2)} \|f\|_{M_1}. \quad (4.52)$$

Moreover, we have

$$2^{jn\alpha_2/2} 2^{jn(\alpha_1 - \alpha_2)(1/q - 1/2)} = 2^{j\widetilde{A}_2(\mathbf{p}, q, \alpha_1, \alpha_2)} = 2^{j\widetilde{A}_3(\mathbf{p}, q, \alpha_1, \alpha_2)} \quad (4.53)$$

in this case.

Combining with the estimates of Case 2.3, Case 2.4, Case 2.5 and Case 2.6, we obtain the upper bound estimates for \widetilde{S}_2 . In addition, we use the estimates of Case 2.2, Case 2.5 and Case 2.6 to deduce

$$\|\square_k^{\alpha_1} | M_1 \rightarrow M_2 \| \lesssim \langle k \rangle^{\frac{\widetilde{A}_3(\mathbf{p}, q, \alpha_1, \alpha_2)}{1 - \alpha_1}}. \quad (4.54)$$

for \widetilde{S}_3 .

□

5. THE EMBEDDING RELATIONS BETWEEN α -MODULATION SPACES

For $0 < p_1, p_2, q \leq \infty$ and $(\alpha_1, \alpha_2) \in [0, 1] \times [0, 1]$, we recall that

$$R(\mathbf{p}, \mathbf{q}; \alpha_1, \alpha_2) = \begin{cases} A(\mathbf{p}, q_1; \alpha_1, \alpha_2), & \text{if } \alpha_1 \leq \alpha_2, \\ A(\mathbf{p}, q_2; \alpha_1, \alpha_2), & \text{if } \alpha_1 \geq \alpha_2. \end{cases}$$

After the preparation in the last two sections, we are now in a position to give the final proof of Theorem 1.2 in the following.

Proof of Theorem 1.2. We only show the proof for $\alpha_1, \alpha_2 < 1$, the other cases can be treated similarly.

Firstly, we claim that $1/p_2 \leq 1/p_1$ is necessary if the embedding relation holds. In fact, we can choose a smooth function h whose Fourier transform \widehat{h} has small compact support, and denote $\widehat{h}_\lambda(\xi) = \widehat{h}(\frac{\xi}{\lambda})$. Then the embedding $M_{p_1, q_1}^{s_1, \alpha_1} \subseteq M_{p_2, q_2}^{s_2, \alpha_2}$ implies

$$\|h_\lambda\|_{L^{p_2}} \lesssim \|h_\lambda\|_{L^{p_1}} \quad (5.1)$$

as $\lambda \rightarrow 0$, which implies $1/p_2 \leq 1/p_1$.

On the other hand, one can easily verify that

$$\|\square_k^{\alpha_2} | M_{p_1, q_1}^{0, \alpha_1} \rightarrow M_{p_2, q_2}^{0, \alpha_2} \| \sim \|\square_k^{\alpha_2} | M_{p_1, q_1}^{0, \alpha_1} \rightarrow M_{p_2, q_1}^{0, \alpha_2} \|.$$

for $\alpha_1 \leq \alpha_2$, and

$$\|\square_k^{\alpha_1} | M_{p_1, q_1}^{0, \alpha_1} \rightarrow M_{p_2, q_2}^{0, \alpha_2} \| \sim \|\square_k^{\alpha_1} | M_{p_1, q_2}^{0, \alpha_1} \rightarrow M_{p_2, q_2}^{0, \alpha_2} \|.$$

for $\alpha_1 \geq \alpha_2$. So Lemma 4.1 implies

$$\|\square_k^{\alpha_1 \vee \alpha_2} | M_{p_1, q_1}^{0, \alpha_1} \rightarrow M_{p_2, q_2}^{0, \alpha_2} \| \sim \langle k \rangle^{\frac{R(\mathbf{p}, \mathbf{q}, \alpha_1, \alpha_2)}{1 - \alpha_1 \vee \alpha_2}}. \quad (5.2)$$

Now, we divide the relations between $1/q_2$ and $1/q_1$ into two cases.

Case One: $1/q_2 \leq 1/q_1$.

In this case, we have $\mathcal{M}_p(l_{q_1}^{s_1, \alpha_1 \vee \alpha_2}, l_{q_2}^{s_2, \alpha_1 \vee \alpha_2}) = l_{\infty}^{s_2 - s_1, \alpha_1 \vee \alpha_2}$ and

$$\|\{ \langle k \rangle^{\frac{R(\mathbf{p}, \mathbf{q}, \alpha_1, \alpha_2)}{1 - \alpha_1 \vee \alpha_2}} \} \|_{l_{\infty}^{s_2 - s_1, \alpha_1 \vee \alpha_2}} = \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{s_2 - s_1}{1 - \alpha_1 \vee \alpha_2}} \langle k \rangle^{\frac{R(\mathbf{p}, \mathbf{q}, \alpha_1, \alpha_2)}{1 - \alpha_1 \vee \alpha_2}}. \quad (5.3)$$

Thus, it is obvious that $s_2 + R(\mathbf{p}, \mathbf{q}, \alpha_1, \alpha_2) \leq s_1$ is the sufficient and necessary condition for the boundedness of (5.3).

Case Two: $1/q_2 > 1/q_1$.

In this case, we have $\mathcal{M}_p(l_{q_1}^{s_1, \alpha_1 \vee \alpha_2}, l_{q_2}^{s_2, \alpha_1 \vee \alpha_2}) = l_r^{s_2 - s_1, \alpha_1 \vee \alpha_2}$, where $1/r = 1/q_2 - 1/q_1$.

$$\|\{ \langle k \rangle^{\frac{R(\mathbf{p}, \mathbf{q}, \alpha_1, \alpha_2)}{1 - \alpha_1 \vee \alpha_2}} \} \|_{l_r^{s_2 - s_1, \alpha_1 \vee \alpha_2}} = \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{r \left[\frac{s_2 - s_1}{1 - \alpha_1 \vee \alpha_2} + \frac{R(\mathbf{p}, \mathbf{q}, \alpha_1, \alpha_2)}{1 - \alpha_1 \vee \alpha_2} \right]} \right)^{1/r}. \quad (5.4)$$

One can easily verify that $s_2 + R(\mathbf{p}, \mathbf{q}, \alpha_1, \alpha_2) + \frac{n(1 - \alpha_1 \vee \alpha_2)}{q_2} < s_1 + \frac{n(1 - \alpha_1 \vee \alpha_2)}{q_1}$ is the sharp condition for the boundedness of (5.4).

We now complete the proof of Theorem 1.2 with the aid of Corollary 3.3.

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